

PERRON-FROBENIUS THEOREM ON NON-NEGATIVE SQUARE MATRICES: AN ELEMENTARY PROOF

By KOTARO SUZUMURA*

I. Introduction

The classical Perron-Frobenius theorem on non-negative square matrices has been of indispensable value in many branches of economic theory. In this brief note we would like to put forward an elementary proof of this well-known theorem, which would be of pedagogical interest.

For any two real $n \times n$ matrices $A=(a_{ij})$ and $B=(b_{ij})$ we denote

$A \geq B$ if and only if $a_{ij} \geq b_{ij}$ ($i, j=1, 2, \dots, n$);

$A \geq B$ if and only if $A \geq B$ and $A \neq B$; and

$A > B$ if and only if $a_{ij} > b_{ij}$ ($i, j=1, 2, \dots, n$).

In particular, a real $n \times n$ matrix $A=(a_{ij})$ is said to be *positive* or *semi-positive* or *non-negative* if and only if $A > 0$ or $A \geq 0$ or $A \geq 0$ holds true, respectively, where 0 denotes an $n \times n$ matrix, all of whose elements are zero. Inequalities between two real vectors with n components and the positivity, semi-positivity and non-negativity of a real vector with n components may be defined in the like manner.

Let $A=(a_{ij})$ be a real $n \times n$ matrix and consider the existence of a scalar λ and a vector with n components $x=(x_i)$ other than 0 such that

$$\lambda x_i = \sum_{j=1}^n a_{ij} x_j \quad (i=1, 2, \dots, n) \quad (1)$$

holds true. Such a scalar λ is called a *characteristic root* of A and a vector x a *characteristic vector* of A associated with λ . It is clear that λ is a characteristic root of A if and only if it satisfies the *characteristic equation* of A , which is defined by

$$\Delta(\lambda) = \det (\lambda I - A) = 0, \quad (2)$$

where I denotes the $n \times n$ identity matrix. $\Delta(\lambda)$ being an n -th order polynomial with real coefficients, the characteristic root of A depends continuously on the elements of A .

The problem of our concern is the special properties which a characteristic root and the associated characteristic vector have when the matrix A is non-negative.

II. Positive Matrices

The first order of business is to prove the following:

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Theorem 1

Let $A=(a_{ij})$ be a positive $n \times n$ matrix. Then:

- (a) A has a positive characteristic value $\lambda(A)$ which is called the Frobenius root of A ;
- (b) A characteristic vector of A associated with $\lambda(A)$ is positive and is unique up to a scalar multiplication; and
- (c) Any characteristic root ω of A , which is complex in general, satisfies $|\omega| \leq \lambda(A)$, where $|\omega|$ denotes the absolute value of ω .

Proof: Identically permuting columns and rows of A if necessary, we may assume without loss of generality that

$$a_{11} \geq a_{22} \geq \dots \geq a_{nn} > 0 \quad (3)$$

holds true. We proceed by induction on the order n of A .

If $n=1$, (1) becomes $\lambda x_1 = a_{11}x_1$ with $a_{11} > 0$. Then the assertions (a) and (b) of the theorem are clearly true.

Assume next that the assertions (a) and (b) of the theorem are true if $n=m$. We now introduce two lemmas which are of use in our proof.

Lemma 1

Let $B=(b_{ij})$ and $C=(c_{ij})$ be two positive $n \times n$ matrices such that $B \geq C$ holds true. Then any characteristic root ω of C satisfies $|\omega| \leq \lambda(B)$.

Proof: Let $B'=(b'_{ij})$ be the transposition of B , i.e. $b'_{ij}=b_{ji}$ ($i, j=1, 2, \dots, n$). Since the characteristic equation of B' is identical with that of B , we obtain $\lambda(B')=\lambda(B)$. Let $x=(x_i)$ be the characteristic vector of B' associated with $\lambda(B)$:

$$\lambda(B)x_i = \sum_{j=1}^n b'_{ij}x_j, \quad x_i > 0 \quad (i=1, 2, \dots, n). \quad (4)$$

Let ω and $y=(y_i)$ be a characteristic root and an associated characteristic vector of C :

$$\omega y_i = \sum_{j=1}^n c_{ij}y_j, \quad y \neq 0 \quad (i=1, 2, \dots, n). \quad (5)$$

Taking the absolute value of the both sides of (5) and letting $y^0=(|y_i|)$, we obtain:

$$|\omega| y_i^0 \leq \sum_{j=1}^n c_{ij} y_j^0 \quad (i=1, 2, \dots, n). \quad (6)$$

Multiplying $x_i > 0$ to the i -th inequality in (6) and adding the outcome over $i=1, 2, \dots, n$, we obtain:

$$|\omega| \sum_{i=1}^n x_i y_i^0 \leq \sum_{i=1}^n \sum_{j=1}^n x_i c_{ij} y_j^0 \leq \sum_{i=1}^n \sum_{j=1}^n x_i b_{ij} y_j^0 = \lambda(B) \sum_{i=1}^n x_i y_i^0,$$

where use is made of $B \geq C > 0$, $b'_{ij}=b_{ji}$ ($i, j=1, 2, \dots, n$) and (4). Since $\sum_{i=1}^n x_i y_i^0 > 0$ is the case, we then obtain $\lambda(B) \geq |\omega|$, as desired. ||

The use of this lemma in what follows is two-fold. Firstly, applying the lemma to $\omega = \lambda(C)$, we see that $B \geq C > 0$ entails $\lambda(B) \geq \lambda(C)$, so that the Frobenius root of a positive matrix is a non-decreasing function of its components. Secondly, applying the lemma to $A=B=C > 0$, we see that the assertion (c) of the theorem is thereby guaranteed if the assertions (a) and (b) are true. Therefore we have only to show that the assertions (a) and (b) are valid for $n=m+1$, given the validity thereof for $n=m$.

Lemma 2

The Frobenius root $\lambda(B)$ of a positive $m \times m$ matrix B satisfies

$$\max_{1 \leq j \leq m} \left(\sum_{i=1}^m b_{ij} \right) \geq \lambda(B) \geq \min_{1 \leq j \leq m} \left(\sum_{i=1}^m b_{ij} \right). \quad (7)$$

Proof: By definition of $\lambda(B)$, we obtain $\lambda(B)x_i = \sum_{j=1}^m b_{ij}x_j$ and $x_i > 0$ for all $i=1, 2, \dots, m$. Upon normalization $\sum_{i=1}^m x_i = 1$ and addition over $i=1, 2, \dots, m$, we obtain $\lambda(B) = \sum_{i=1}^m \sum_{j=1}^m b_{ij}x_j = \sum_{j=1}^m \left(\sum_{i=1}^m b_{ij} \right) x_j$. Since $x_j > 0$ ($j=1, 2, \dots, m$) and $\sum_{j=1}^m x_j = 1$ are true, (7) obtains therefrom. ||

Coming back to the proof of the theorem, we now take any scalar $\theta > 0$ and consider the following system:

$$\lambda x_i = \sum_{j=1}^m (a_{ij} + \theta a_{i,m+1} a_{m+1,j}) x_j \quad (i=1, 2, \dots, m). \quad (8)$$

Since $a_{ij} + \theta a_{i,m+1} a_{m+1,j} > 0$ is true for all $i, j=1, 2, \dots, m$, our induction hypothesis ensures the existence of a scalar $\lambda(\theta) > 0$ and a vector $x(\theta) = (x_i(\theta))$ with $x_i(\theta) > 0$ for all $i=1, 2, \dots, m$. Furthermore $x(\theta)$ is unique up to a scalar multiplication, so that we may normalize it by requiring $\sum_{i=1}^m x_i(\theta) = 1$. We may thereby claim the *unique* existence of $x(\theta)$ in the relative interior of the fundamental m -simplex:

$$S_m = \{x = (x_i) \mid x_i \geq 0 \ (i=1, 2, \dots, m) \text{ and } \sum_{i=1}^m x_i = 1\}. \quad (9)$$

Define $x_{m+1}(\theta) > 0$ by

$$x_{m+1}(\theta) = \theta \sum_{j=1}^m a_{m+1,j} x_j(\theta). \quad (10)$$

Then we obtain:

$$\sum_{j=1}^{m+1} a_{ij} x_j(\theta) = \sum_{j=1}^m (a_{ij} + \theta a_{i,m+1} a_{m+1,j}) x_j(\theta) \quad (11)$$

for all $i=1, 2, \dots, m$. Furthermore we have:

$$\sum_{j=1}^{m+1} a_{m+1,j} x_j(\theta) = (\theta^{-1} + a_{m+1,m+1}) x_{m+1}(\theta). \quad (12)$$

In view of (8), (11) and (12), we are home if we may prove the unique existence of a scalar $\theta^* > 0$ such that $\theta^{*-1} + a_{m+1,m+1} = \lambda(\theta^*)$ holds true. Note that a function $\xi(\theta) = \theta^{-1} + a_{m+1,m+1}$ of θ is a hyperbola which approaches $+\infty$ and $a_{m+1,m+1}$ as θ tends to 0 and $+\infty$, respectively, while $\lambda(\theta)$ is a continuous non-decreasing function of θ which starts from $\lambda(0) > a_{m+1,m+1}$ by virtue of (3), Lemma 1 and Lemma 2. Therefore $\xi(\theta)$ and $\lambda(\theta)$ cross just once, which concludes our proof. ||

III. Non-Negative Matrices

Consider now any non-negative $n \times n$ matrix $A = (a_{ij})$. For any positive scalar $\varepsilon > 0$ we may define a positive $n \times n$ matrix $A(\varepsilon) = (a_{ij}(\varepsilon))$ by

$$a_{ij}(\varepsilon) = \begin{cases} a_{ij} & \text{if } a_{ij} > 0 \\ \varepsilon & \text{if } a_{ij} = 0. \end{cases}$$

We may now invoke Theorem 1 to assert the existence of $\lambda(A(\varepsilon)) > 0$ and the unique positive vector $x(\varepsilon) \in S_n$ which is associated with $\lambda(A(\varepsilon))$. Noticing this fact we take a sequence of positive scalars $\{\varepsilon_\nu\}_{\nu=1}^\infty$ such that $\varepsilon_\nu > \varepsilon_{\nu+1}$ for all ν and $\lim_{\nu \rightarrow \infty} \varepsilon_\nu = 0$. We then obtain a sequence $\{\lambda(A(\varepsilon_\nu))\}_{\nu=1}^\infty$ and $\{x(\varepsilon_\nu)\}_{\nu=1}^\infty$ such that

$$\lambda(A(\varepsilon_\nu)) > 0 \text{ and } 0 < x(\varepsilon_\nu) \in S_n \text{ for all } \nu = 1, 2, \dots \quad (13)$$

and

$$\lambda(A(\varepsilon_\nu))x_i(\varepsilon_\nu) = \sum_{j=1}^n a_{ij}(\varepsilon_\nu)x_j(\varepsilon_\nu) \quad (i=1, 2, \dots, n). \quad (14)$$

S_n being compact we may choose an appropriate subsequence $\{\varepsilon_{\nu(\mu)}\}_{\mu=1}^\infty$ of $\{\varepsilon_\nu\}_{\nu=1}^\infty$ such that $\{x(\varepsilon_{\nu(\mu)})\}_{\mu=1}^\infty$ converges. Let $x(0) = \lim_{\mu \rightarrow \infty} x(\varepsilon_{\nu(\mu)}) \in S_n$. Since the Frobenius root of A depends continuously on the components of A , we obtain $\lim_{\mu \rightarrow \infty} \lambda(A(\varepsilon_{\nu(\mu)})) = \lambda(A(\lim_{\mu \rightarrow \infty} \varepsilon_{\nu(\mu)})) = \lambda(A)$. It then follows from (13) and (14) that:

$$\lambda(A) \geq 0, x(0) \in S_n, \lambda(A)x_i(0) = \sum_{j=1}^n a_{ij}x_j(0) \quad (i=1, 2, \dots, n). \quad (15)$$

Let ω be any characteristic root of A . If it so happens that $|\omega| > \lambda(A)$ holds true, then there exists a sufficiently small $\varepsilon > 0$ such that $|\omega(\varepsilon)| > \lambda(A(\varepsilon))$ by continuity, where $\omega(\varepsilon)$ is a characteristic root of $A(\varepsilon)$. But this contradicts Theorem 1(c). Therefore we obtain the following:

Theorem 2

Let $A = (a_{ij})$ be any non-negative $n \times n$ matrix. Then:

- (a) A has a non-negative characteristic root $\lambda(A)$;
- (b) A characteristic vector of A associated with $\lambda(A)$ is semi-positive; and
- (c) For any characteristic root ω of A , $|\omega| \leq \lambda(A)$ holds true.

IV. Non-Negative Indecomposable Matrices

Note that Theorem 2 asserts somewhat *weaker* properties for a *wider* class of non-negative square matrices than Theorem 1, which is concerned only with positive square matrices. What is lost in the passage from Theorem 1 to Theorem 2 may be recovered, however, if a non-negative matrix in question is indecomposable in the sense specified below.

Let $A = (a_{ij})$ be a non-negative $n \times n$ matrix. If there exists a partition (I, J) of $\{1, 2, \dots, n\}$ such that

$$I \cap J = \emptyset, I \cup J = \{1, 2, \dots, n\}, I \neq \emptyset, J \neq \emptyset, a_{ij} = 0 \text{ for all } (i, j) \in I \times J,$$

then A is *decomposable*. Otherwise A is *indecomposable*.

We may then assert the following proposition. As a matter of fact, Theorem 3 subsumes Theorem 1 since any positive matrix is trivially nonnegative indecomposable matrix.

Theorem 3

Let $A = (a_{ij})$ be a non-negative indecomposable $n \times n$ matrix. Then:

- (a) A has a positive characteristic root $\lambda(A)$;
- (b) A characteristic vector of A associated with $\lambda(A)$ is positive and is unique up to a scalar multiplication; and
- (c) For any characteristic root ω of A , $|\omega| \leq \lambda(A)$ holds true.

Proof: Thanks to Theorem 2, A has a characteristic root $\lambda(A) \geq 0$ with which a semi-positive characteristic vector x is associated.

Suppose that $x_i = 0$ for all $i \in I$ and $x_i > 0$ for all $i \in I^c = \{1, 2, \dots, n\} \setminus I$. It then follows that

$$\sum_{j \in I^c} a_{ij} x_j = \lambda(A) x_i = 0 \text{ for all } i \in I \quad (16)$$

holds true. Since $x_j > 0$ for all $j \in I^c$, we obtain $a_{ij} = 0$ for all $(i, j) \in I \times I^c$, which is a contradiction unless $I = \phi$. Therefore $x > 0$ must be the case.

If $\lambda(A) = 0$ is true, then $\sum_{j=1}^n a_{ij} x_j = 0$ for all $i = 1, 2, \dots, n$. Coupled with $x > 0$ we then obtain $A = 0$, which contradicts indecomposability. Therefore $\lambda(A) > 0$ must be the case.

Finally, suppose that there are two characteristic vectors x and y which associate with $\lambda(A)$. Since $x, y > 0$ is true, $\eta = \min_{1 \leq i \leq n} (x_i / y_i)$ is well-defined. Let $z = x - \eta y$. Then z is non-negative with at least one zero component. Assume that z is non-zero. We are assured that

$$\sum_{j=1}^n a_{ij} z_j = \sum_{j=1}^n a_{ij} (x_j - \eta y_j) = \lambda(A) (x_i - \eta y_i) = \lambda(A) z_i$$

holds true for all $i = 1, 2, \dots, n$, so that z qualifies as a characteristic vector associated with $\lambda(A)$. But this is a contradiction as z contains at least one zero component. Therefore $z = 0$, i.e. $x = \eta y$ must be the case, which completes the proof of the theorem. ||

HITOTSUBASHI UNIVERSITY

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